An exact-order estimate of the norms of orthogonal projection operators onto spaces of continuous splines[1]

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Notation and formulation of the problem

We introduce the following notation. Fix a positive integer *k* and an interval [*a*,*b*]. Let $\Delta_n = \{a = x_0 < x_1 < \ldots < x_{n-1} < x_n = b\}$ denote a partition of the interval [*a*,*b*], $\mathbb{P}_k$ – the space of polynomials of degree no greater than *k*. Then the space

$$\mathbb{S}_{k,m}(\Delta_n) = \left\{ s \in C^{k-m}[a,b] : s\big|_{[x_i, x_{i+1}]} \in \mathbb{P}_k, i = 0, \ldots, n-1 \right\}$$

is called the *space of polynomial splines of degree k and defect m with knot sequence* Δ_n (where $1 \le m \le k+1$). When $m=k+1$ (*maximal defect splines*), there are no continuity conditions imposed upon the spline at the knots, such a spline is of a totally local nature. When $m=k$ (*continuous splines*, or *C-splines*), one continuity condition is imposed upon the spline at each knot. When $m=1$ (*minimal defect splines*), the spline and all of its derivatives up to order k-1 are continuous at the knots and only the last derivative, of order *k*, can have discontinuities at the knots.

Consider the operator $P_{\mathbb{S}}$ of orthogonal (with respect to the inner product in the space $L_2[a,b]$) projection of the space $C[a,b]$ onto its subspace $\mathbb{S}_{k,m}(\Delta_n)$:

$$P_{\mathbb{S}} : C[a,b] \ni f \mapsto P_{\mathbb{S}} f \in \mathbb{S}_{k,m}(\Delta_n)$$

where $P_{\mathbb{S}} f$ is characterized by either of the following two equivalent conditions:

1. $\langle f, s \rangle = \langle P_{\mathbb{S}} f, s \rangle$ for all $s \in \mathbb{S}_{k,m}(\Delta_n)$,
2. $\| f - P_{\mathbb{S}} f \|_2 = \min_{s \in \mathbb{S}_{k,m}(\Delta_n)} \| f - s \|_2$.

Consider the norm of the operator $P_{\mathbb{S}}$ as an operator from $C[a,b]$ to $C[a,b]$. It is rather easy to verify that when partition Δ_n is fixed, the operator $P_{\mathbb{S}}$ is bounded. It is therefore natural to pose the following question: is the family of projection operators $P_{\mathbb{S}}$ bounded *uniformly in all admissible partitions* Δ_n *of the interval* [*a*,*b*]?

Denote

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$$c_{k,m} = \sup_{\Delta_n, n} \left\| P_{\mathbb{S}_{k,m}(\Delta_n)} \right\|_\infty$$

The numbers $c_{k,m}$ are frequently referred to as *Lebesgue constants*. The general problem is to establish whether these quantities are finite and (in case of their finiteness) to calculate them for various $k\geq1$, $1\leq m\leq k$. (The case $m=k+1$ does not belong to spline theory proper.) In this paper we address the problem of estimating Lebesgue constants of continuous splines, i.e. the numbers $c_{k,k}$ with $k\geq2$.

History of the problem

The conjecture that $c_{k,1}<\infty$ for all $k\geq1$ was stated by Carl de Boor in [2], that is why the question of its validity bears (or, rather, bore until recently) the name of *de Boor's problem* or *de Boor's conjecture*. (Note that higher-defect splines can be obtained from minimal defect splines by a limiting process, which also provides the estimate $c_{k,m} \leq c_{k,1}$.) The following table summarizes the history of de Boor's problem for minimal defect splines.

Splines	Author	Year	Result
k=1 (linear)	Z. Ciesielsky [5]	1963	$c_{1,1}\leq3$
k=1 (linear)	Z. Ciesielsky [6]	1975	Problem: does $c_{1,1}=3$ hold?
k=1 (linear)	K.I. Oskolkov [7] and P. Oswald [11] (independently)	1977	$c_{1,1}=3$
k=2 (parabolic)	C. de Boor [1]	1968	$c_{2,1}\leq30$
k=3 (cubic)	C. de Boor [4]	1979	$c_{3,1}\leq245/3$
any k	A.Yu. Shadrin [9]	2001	$c_{k,1}<\infty$ $c_{k,1}\geq2k+1$ Conjecture: $c_{k,1}=2k+1$.

The operator of projection onto spaces of higher-defect splines has also been examined in the literature.

1. De Boor in [3] (1976) proved that $c_{k,k} <\infty$. As remarked in [9], from the results of [3] one can derive the estimate $c_{k,k} \leq 4^k/\sqrt{k}$.

2. From the results of N.L. Zmatrakov and Yu.N. Subbotin contained in [10] (1983) it also follows that $c_{k,k} <\infty$.

3. A.Yu. Shadrin in [8] (1998) proved that $c_{k,m} < \infty$ for $k \geq 5$, $m = k-1$ and $k \geq 10$, $m = k-2$, as well as for $k = 17, 18$, $m = k-3$ and $k = 26, 27$, $m = k-4$.

But no constructive upper bounds for $c_{k,m}$, apart from the obvious bounds $c_{2,2} \leq c_{2,1} \leq 30$ and $c_{3,3} \leq c_{3,1} \leq 245/3$, have thus far been obtained.

In summary, Lebesgue constants $c_{k,m}$ have been proven to be finite for all $k \geq 1$ and $1 \leq m \leq k+1$, but constructive upper bounds exist only for $k = 1,2,3$. Therefore, the problem of deriving upper estimates for the numbers $c_{k,m}$ for various k and m is currently on the agenda.

Importance of the problem

This problem is interesting for several reasons.

1. Studying the norms of orthogonal projection operators onto various approximating spaces (i.e. their Lebesgue constants) is a classical problem in approximation theory. Their behavior has been profoundly studied in the case of projection with various weights onto spaces of algebraic and trigonometric polynomials. As described above, in spline theory there still remain many unresolved problems in this direction.

2. The operator of orthogonal projection onto splines of degree k is closely related to the operator of interpolation by splines of degree $2k+1$. In fact, the study of Lebesgue constants was initiated by Carl de Boor with the aim of deriving new results for interpolation operators. Interpolating splines (with non-uniform knots) are extensively used both for theoretical purposes and for the solution of practical approximation problems of all kinds. The derivation of new estimates for the orthogonal projection operator automatically leads to new results on the interpolation operator, – which, in particular, allow one to estimate the effect of errors in initial data on the error of approximation.

3. Finally, constructive estimates for Lebesgue constants can be useful in deriving and studying the behavior of various practical algorithms of approximation by splines.

The result

Theorem 1. *The following estimate holds for Lebesgue constants $c_{k,k}$ of continuous splines of degree k:*

$$\mu_k \leq c_{k,k} \leq A\mu_k \,,$$

where

$$\mu_k = \frac{(k+1)(k+2)}{2}\int_{-1}^{1}\left|I_k(t)\right|dt,$$

$$I_k(t) = \frac{1+t}{2} - \frac{2}{3}(1-t^2)\sum_{j=0}^{k-2}\frac{1}{h_j}\ C_j^{5/2}(t) =$$

$$= \frac{3}{k(k+1)(k+2)}(1+t)\left(C_{k-1}^{5/2}(t) - C_{k-2}^{5/2}(t)\right) = \frac{1}{2k}(1+t)P_{k-1}^{(1,2)}(t).$$

Here $C_k^{\lambda}(t)$ is the Gegenbauer polynomial of order λ and degree k, h_j is the square of its L_2-norm, $P_k^{(\alpha,\beta)}(t)$ is the Jacobi polynomial with parameters (α,β) and of degree k. Concerning the value of the constant A (which is independent of k) see lemma 4.

Theorem 2. *The exact order of growth of Lebesgue constants $c_{k,k}$ is $k^{1/2}$.*

Note that this result was obtained with the aid of the following representation of elements of the space $\mathbb{S}_{k,k}(\Delta_n)$ of continuous splines:

$$s(x) = \sum_{l=0}^{n} z_l B_l(x) + \sum_{i=0}^{n-1}\chi_i(x)(x-x_{i+1})(x-x_i)\sum_{j=0}^{k-2} c_j^i C_j^{5/2}\left(\frac{2(x-x_i)}{x_{i+1}-x_i}-1\right),$$

where

$\left(B_l(x)\right)_{l=0}^{n}$ is the B-spline basis in the space of linear splines on partition Δ_n,

$\chi_i(x)$ is the characteristic function of the i^{th} subinterval.

The proof makes extensive use of the theory of classical orthogonal polynomials. Below are the principal lemmas of the proof.

Lemma 1. *Row scaling can transform the matrix of normal equations for the determination of coefficients $\left(z_l\right)_{l=0}^{n}$ to a row-diagonally dominant matrix with the coefficient of diagonal dominance equaling $\frac{1}{(k+1)(k+2)}$.*

Lemma 2. *The following estimate holds for coefficients $\left(z_l\right)_{l=0}^{n}$:*

$$\max_{l=0,\dots,n}\left|z_l\right| \le \mu_k .$$

Lemma 3. *The following estimate holds for the Lebesgue function of continuous splines on the subinterval $[x_i,x_{i+1}]$ (after the linear change of variable mapping $[x_i,x_{i+1}]$ to $[-1,1]$):*

$$\sup_{\|f\|_\infty\le 1}\left|\left(P_{\mathbb{S}}f\right)(t)\right| \le \mu_k\left(\left|I(t)\right| + \left|I(-t)\right|\right) + \int_{-1}^{1}(1-t^2)(1-\tau^2)\left|\sum_{j=0}^{k-2}\frac{1}{h_j}\ C_j^{5/2}(t)C_j^{5/2}(\tau)\right| d\tau .$$

Lemma 4. *There exists a number A such that*

$$\mu_k\left(\left|I(t)\right| + \left|I(-t)\right|\right) + \int_{-1}^{1}(1-t^2)(1-\tau^2)\left|\sum_{j=0}^{k-2}\frac{1}{h_j}\ C_j^{5/2}(t)C_j^{5/2}(\tau)\right| d\tau \le A\mu_k .$$

This inequality is valid with A=2. For k=2 and 3 it is valid with A=1.

Numerical calculations show that the inequality is valid with A=1 for all *k* from 4 to 30. The conjecture that it is valid with A=1 for all values of *k* is now being investigated.

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